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The Calculation of Large Reynolds Number Two-Dimensional Flow Using Discrete Vortices with Random Walk

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The numerical calculation of two-dimensional rotational flow at large Reynolds number is considered. The method of replacing a continuous distribution of vorticity by a finite number, N, of discrete vortices is examined, where the vortices move under their mutually induced velocities plus a random component to simulate effects of viscosity. The accuracy of the method is studied by comparison with the exact solution for the decay of a circular vortex. It is found, and analytical arguments are produced in support, that the quantitative error is significant unless N is large compared with a characteristic Reynolds number. The mutually induced velocities are calculated by both direct summation and by the "cloud in cell" technique. The latter method is found to produce comparable error and to be much faster.

1. INTRODUCTION

Molecular transport processes in a gas, which lead to the diffusion of mass and momentum, are due to the random motion of the molecules superposed on the mean motion of the fluid. The effects of diffusion or viscosity can therefore be studied by adding, to the continuum velocity of a fluid particle, a random component which describes the Brownian motion of the molecules. If $\mathbf{u}(\mathbf{x}, t)$ is the continuum velocity field, $\mathbf{r}(t)$ is the position of a fluid particle, and ν is the diffusivity, we consider the equation

$$(d/dt)\,\mathbf{r}(t) = \mathbf{u}(\mathbf{x}, t) + \mathbf{w}(t), \tag{1}$$

where $\mathbf{w}$ is a stationary random function of time with the property that

$$\langle w_i(t)\, w_j(t')\rangle = 2\nu R(t - t')\,\delta_{ij}\,, \tag{2}$$

where

$$\int_0^\infty R(\tau)\, d\tau = 1. \tag{3}$$

This idea is an obvious extension of the well-known concept that the diffusion equation is equivalent to a random walk. Saffman [1, 2, 3] used it to study the effects of mole-

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cular diffusion on the transport of a passive scalar in flow through porous media and in homogeneous and inhomogeneous turbulent flow. It is to be noted that although the original physical justification of the idea was based on the gaseous structure of the fluid, the fact that the results involve only macroscopic properties of the fluid implies that they apply equally well to liquids.

In 1969, Professor D. W. Moore pointed out in a private communication that, since two-dimensional inviscid flow of an incompressible fluid can be modeled by following discrete line vortices which move with the fluid, viscous effects could be incorporated by adding a random walk component to the displacement of the of the vortices. In unbounded fluid, the velocity of each vortex would be supposed to be given by (1), where $\mathbf{u}$ is the velocity induced by all other vortices. Professor Moore and one of us (P.G.S.) tested the idea by using 50 discrete vortices to approximate the viscous decay of an initially uniform rectilinear vortex of circular cross section, for which a simple exact solution exists. The agreement was found to be poor; so bad, in fact, that the method was abandoned without further study. It was realized that the error would be reduced by increasing the number of vortices, N. However, standard results suggested that the decrease is like $N^{-1/2}$ and since the computing time increases like N^2 better calculations were prohibitively expensive.

Subsequently, the method was presented by Chorin [4] and applied by him to the study of flow past a circular cylinder for Reynolds numbers from 100 to over 10,000. This requires two extensions of the basic idea. First, image vortices must be created (or more generally, a Green's function has to be introduced) to satisfy the zero normal velocity kinematic requirement at the solid surface. Second, new vortices have to be created at the surface at a rate governed by the need to satisfy the no-slip boundary condition (see [5]). Chorin and others [6] have applied the method to a variety of two-dimensional flows and the idea has attracted much attention. Since none of the results can be compared with exact solutions, and a priori error estimates have not been obtained, the reliability of the method is open to question.[1] Comparison with gross experimental features, such as drag, is not conclusive, and in any case the real flows are turbulent, and therefore three dimensional, at large Reynolds number. Moreover, one must distinguish between a method to model turbulent flow and a method to approximate an exactly two-dimensional flow.

There are also uncertainties in the method. For numerical purposes it is necessary to approximate the velocity field of a line vortex, which is singular at the vortex itself, by that of a vortex of finite cross section. The flow fields of vortices of finite cross section cause deformations of the cores, where in general the larger the core size the greater is the deformation, and the neglect of this effect poses unresolved questions.

It was therefore decided to re-examine the problem studied in 1969 by Moore and

[1] After the work of the present paper was completed, Dr. W. Ashurst of Sandia Laboratories, Livermore, told us of his calculation to check the method by studying the Rayleigh problem of the boundary layer growth on a flat plate set impulsively in motion, for which an exact solution is known. It appears that good agreement can be found, but the results are sensitive to the way in which vorticity is introduced at the walls.

Saffman, of the decay of a single vortex of finite size, but now to use 1000 vortices to investigate the accuracy of the approach. This problem has the advantage that it is an unbounded flow and so the complications and further doubts occasioned by the presence of solid walls are avoided. The method of calculation and results will be presented in the following sections. One would like to have done more than was actually done (for instance, ensemble averages over initial conditions are desirable), but the present results took about 8 hours on an IBM 370/158 and exhausted our resources. However, we think that the evidence is sufficient to support a claim that uncritical use may lead to significant error. We shall also present some simple analysis in support of the belief.

2. The Calculation

We take N line vortices, the ith vortex having position (x_i, y_i) and strength κ_i. The equation of motion (1) becomes

$$\frac{dx_i}{dt} = -\frac{1}{\kappa_i}\frac{\partial H}{\partial y_i} + w_{x_i}, \qquad \frac{dy_i}{dt} = \frac{1}{\kappa_i}\frac{\partial H}{\partial x_i} + w_{y_i}, \tag{4}$$

where

$$H = \frac{1}{2\pi}\sum_{i=1}^{N}{\sum_{j=1}^{N}}' \kappa_i\kappa_j \log r_{ij}. \tag{5}$$

The prime indicates that $i = j$ is excluded from the sum, and

$$r_{ij} = [(x_i - x_j)^2 + (y_i - y_j)^2]^{1/2}. \tag{6}$$

The ensemble average of the solution of (4) and (5), over the white noise $\mathbf{w}$, gives a solution of the Navier–Stokes equations for the initial condition of N line vortices at the points $x_i(0) = x_i^0$, $y_i(0) = y_i^0$. We speculate (but have not proved) that a further ensemble average over random initial positions, (x_i^0, y_i^0), of the line vortices will give a solution of the Navier–Stokes equations for an initially continuous distribution.

For inviscid flow, H is an invariant of the motion. The linear and angular impulses, given, respectively, by

$$I_x = \sum_{i=1}^{N} \kappa_i y_i, \qquad I_y = -\sum_{i=1}^{N} \kappa_i x_i, \qquad \bar{A} = -\tfrac{1}{2}\sum_{i=1}^{N} \kappa_i(x_i^2 + y_i^2), \tag{7}$$

are also invariant if no net force or couple is applied over boundary walls; this is the case if the fluid is unbounded. In a viscous unbounded fluid, the linear impulse is invariant but the angular impulse changes with time. The square of the radius of gyration $A = -2\bar{A}/\Gamma$ grows like $4\nu t$, where $\Gamma = \sum_{i=1}^{N} \kappa_i$ is of course invariant.

When vortices are close together, the inviscid velocities become very large and for numerical reasons it is advisable to replace the singular velocity distribution by one which gives finite velocities as $r_{ij} \to 0$. Thus, we make the approximation that

$$H = \frac{1}{2\pi} \sum_{i=1}^{N} \sum_{j=1}^{N}{}' \kappa_i \kappa_j F(r_{ij}), \tag{8}$$

where the function F is chosen in some ad hoc manner. We shall take

$$F(\xi) = \begin{cases} \log(\xi/\delta), & \xi > \delta, \quad (9) \\ \frac{1}{2}\xi^2/\delta^2, & \xi < \delta. \quad (10) \end{cases}$$

The value of δ will be discussed below. Chorin [4] replaced (10) by $F(\xi) = \xi/\delta$, with $2\pi\delta$ equal to the spacing between the new vortices created at the walls to satisfy the no-slip boundary condition. Kuwahara and Takami [7] took

$$F(\xi) = (1/\xi)(1 - e^{-\xi^2/4\nu t}). \tag{11}$$

Since vortices of finite size deform when they approach too closely, no approach is exact, and a priori error estimates are unknown. The choice of F has no effect on the existence of the invariants.

To integrate (4), we must replace it by a finite difference approximation. Our procedure is to integrate the hydrodynamic part for a time step Δt and then add a random displacement. Specifically, we take

$$\Delta x_i = x_i(t + \Delta t) - x_i(t) = -\frac{1}{\kappa_i} \int_t^{t+\Delta t} \frac{\partial H}{\partial y_i}\, dt + \lambda \left(\zeta_i - \frac{1}{N} \sum_{j=1}^{N} \zeta_j \right), \tag{12}$$

$$\Delta y_i = y_i(t + \Delta t) - y_i(t) = \frac{1}{\kappa_i} \int_t^{t+\Delta t} \frac{\partial H}{\partial x_i}\, dt + \lambda \left(\eta_i - \frac{1}{N} \sum_{j=1}^{N} \eta_j \right). \tag{13}$$

The ζ_i, η_i are independent, Gaussian, random numbers, with zero mean and unit variance. The average is subtracted from the random displacements to ensure that the linear impulse remains constant, or equivalently that the centroid of the vortices is fixed. The variation of computed impulse is then a measure of the error.

The integral is evaluated by modified Euler (also known as Huen's method), i.e.,

$$\begin{aligned} &\int_t^{t+\Delta t} \mathbf{g}(\mathbf{y}, t)\, dt \doteq (\Delta t/2)\{\mathbf{g}(\mathbf{y}(t), t) + \mathbf{g}(\hat{\mathbf{y}}(t + \Delta t), t + \Delta t)\}, \\ &\hat{\mathbf{y}}(t + \Delta t) = \mathbf{y}(t) + \Delta t\, \mathbf{g}(\mathbf{y}(t), t). \end{aligned} \tag{14}$$

Using this scheme the (exactly constant) angular momentum of a pair of line vortices grows as $(\Delta t)^4$, while it grows as $(\Delta t)^2$ if Euler integration is used.

The quantity λ is chosen to ensure that the noise ensemble averaged angular momentum grows at the correct rate. It is easily shown that

$$\lambda = \left(\frac{2\nu\,\Delta t}{(1 - 1/N)}\right)^{1/2}. \tag{15}$$

The value of δ was chosen so that the maximum displacement of a vortex during one time step was less than the average separation of the vortices. For N vortices in a circle of radius R_0, the average separation is $(\pi R_0^2/N)^{1/2}$. At separation δ, the displacement in time step Δt is of the order $(\kappa/2\pi\delta)\,\Delta t$. Thus we set

$$\delta = \frac{\kappa\,\Delta t}{2\pi R_0}\left(\frac{N}{\pi}\right)^{1/2}. \tag{16}$$

We used N vortices of equal strength κ to represent the single vortex of initial radius R_0, strength $\Gamma = \kappa N$, and uniform vorticity. The vortices were placed at random in the initial circle, for the reasons given after Eq. (4). The initial rotation time of the vortex is

$$T = 4\pi^2 R_0^2/\Gamma. \tag{17}$$

Three calculations were done with the same initial conditions: (i) pure hydrodynamics, $\nu = 0$; (ii) pure diffusion, $\kappa = 0$; (iii) hydrodynamics plus diffusion.

The value of ν was chosen so that the appropriate Reynolds number, $Re = \Gamma/\nu$, say, was 2000π. Computations were done with $N = 1000$ and $N = 50$. In practice the normalization employed put $\nu = 1$, $R_0 = 1$, $T = 4\pi^2/2000\pi = \pi/500$.

The time step was chosen so that the change in angular momentum, A, due to numerical error was not more than 1 % of the effect that was to be calculated. It was found that when

$$\Delta t = T/100, \tag{18}$$

the change in A was less than 1 % of $4\nu t$ and the linear impulses were constant to 1 part in 10^{15}. All calculations were carried out in double precision on an IBM 370/158. With this value of Δt, δ is 1/50th of the average separation, and is considerably smaller than the value employed by others. We believe that δ should be as small as possible, consistent with numerical accuracy, in order to model continuous solutions of the Navier–Stokes equations, but there is no hard evidence on this matter and the question is open. In any event, the analysis of Section 4 suggests that our present conclusions do not depend on the value of δ.

3. Results

Figure 1 shows A as a function of t/T for cases (ii) and (iii) with $N = 50$ and $N = 1000$, respectively. The exact solution, $A = A_0 + 4\nu t$, is also shown.

The integration time is $3T$. After three rotations, the relative error

$$\epsilon = (A - A_0 - 4\nu t)/4\nu t,$$

which is shown in Fig. 2, is about 20 % for $N = 50$ and 10 % for $N = 1000$. Over one rotation the relative error is much larger. It can be seen from Fig. 2 that the relative error with $N = 50$ is 2–4 times larger than the corresponding error with $N = 1000$, which is consistent with the belief that the error should behave like $N^{-1/2}$.

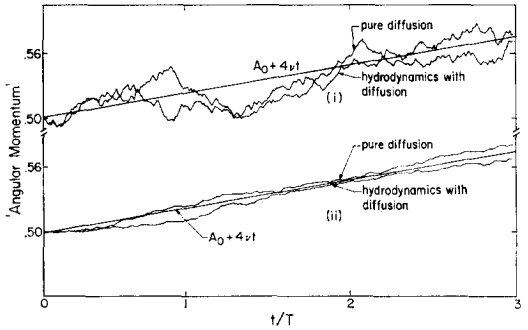

FIG. 1. Random values of A for the pure diffusion of N vortices originally inside a circle. Results are shown for pure diffusion and hydrodynamic motion with diffusion: (i) $N = 50$; (ii) $N = 1000$. The straight line is the exact result.

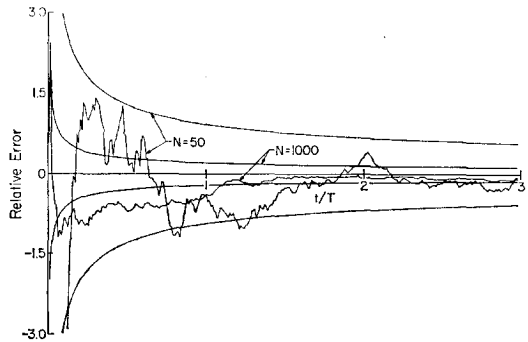

FIG. 2. Relative error vs t ($t < 3T$) for hydrodynamic motion with diffusion $N = 50$, 1000. Also shown is the corresponding standard deviation $\sigma(\epsilon)$ as given by Eq. (30).

The size of the error for $N = 1000$ is surprisingly large, and it was therefore decided to undertake some analysis to try and check the results. This is described in the next section.

4. Analysis

Consider the case of pure diffusion. Let x_i^p, y_i^p be the coordinates of the ith vortex after p time steps, and ζ_i^p, η_i^p the random variables entering at the pth step. Then

$$\begin{aligned} x_i^p &= x_i^{p-1} + \lambda\left(\zeta_i^p - (1/N)\sum_{j=1}^{N}\zeta_j^p\right), \\ y_i^p &= y_i^{p-1} + \lambda\left(\eta_i^p - (1/N)\sum_{j=1}^{N}\eta_j^p\right), \end{aligned} \tag{19}$$

where λ is given by (15). Henceforth, for simplicity we neglect the centroid correction term; it is not hard to see that this is an $O(N^{-1})$ effect in the analysis. Then

$$x_i^p = x_i^0 + \lambda\theta, \qquad y_i^p = y_i^0 + \lambda\phi, \tag{20}$$

where θ and ϕ are normally distributed random variables with

$$E(\theta) = 0, \qquad E(\theta^2) = p, \qquad E(\theta^4) = 6p^2 - 3p, \tag{21}$$

where E denotes expected value, and the same results hold for ϕ. Note that

$$p = t/\Delta t = 2\nu t/\lambda^2, \tag{22}$$

neglecting terms $O(N^{-1})$.

Now

$$A^p \equiv (1/N)\sum_{i=1}^{N}[(x_i^p)^2 + (y_i^p)^2]. \tag{23}$$

Since

$$\begin{aligned} E((x_i^p)^2) &= (x_i^0)^2 + \lambda^2 E(\theta^2), \\ E((y_i^p)^2) &= (y_i^0)^2 + \lambda^2 E(\phi^2), \end{aligned} \tag{24}$$

it follows that

$$E(A^p) = A^0 + 2\lambda^2 p = A^0 + 4\nu t. \tag{25}$$

Thus the expected value of the angular momentum for the N vortices grows at the same rate that it does for the continuous distribution of vorticity which they model.

The order of magnitude of the error in one realization of the flow will be given by the standard deviation of A^p about its expected value.

This is found by calculating the expectation of $(A^p)^2$. A calculation shows that

$$E((x_i^p)^4) = (x_i^0)^4 + 6\lambda^2 p(x_i^0) + \lambda^4(6p^2 - 3p), \tag{26}$$

$$E((x_i^p x_j^p)^2) = (x_i^0 x_j^0)^2 + \lambda^2 p((x_i^0)^2 + (x_j^0)^2) + \lambda^4 p^2, \tag{27}$$

and similarly for y_i^p.

Since

$$(A^p)^2 = (1/N^2) \sum_{i=1}^{N} \{(x_i^p)^4 + (y_i^p)^4 + 2(x_i^p y_i^p)^2\}$$

$$+ (1/N^2) \sum_{i=1}^{N} \sum_{j=1}^{N}{}' \{(x_i^p x_j^p)^2 + (y_i^p y_j^p)^2 + 2(x_i^p y_j^p)^2\}, \tag{28}$$

a straightforward but tedious calculation gives

$$\mathrm{var}(A_\nu^p) = E((A_\nu^p)^2) - [E(A_\nu^p)]^2 = \frac{4\lambda^2 p}{N} A^0 + \frac{2\lambda^4 p^2}{N}\left(5 - \frac{3}{p}\right), \tag{29}$$

where the suffix ν refers to the fact that the result is for pure diffusion. Hence, the standard deviation of the relative error ϵ is given by

$$\sigma(\epsilon) = \frac{1}{N^{1/2}}\left(\frac{A^0}{2\nu t} + \frac{5}{2} - \frac{3}{2p}\right)^{1/2}. \tag{30}$$

Thus for the error to be small it is not sufficient for $N^{1/2} \gg 1$, but it is necessary that

$$N^{1/2} \gg \left(\frac{A^0}{2\nu t}\right)^{1/2}. \tag{31}$$

Since $A^0/2\nu = \Gamma T/16\pi^2\nu$, condition (31) can be expressed as

$$N^{1/2} \gg \frac{Re^{1/2}}{4\pi}\left(\frac{T}{t}\right)^{1/2}. \tag{32}$$

Thus the larger the Reynolds number, the greater must be the number of vortices if the effects of viscosity are to be correctly described.

This analysis is for pure diffusion. Let us now consider the case of hydrodynamics plus diffusion. An exact analysis cannot now be done, but the following discussion makes plausible the conclusion that the error is not less than that for pure diffusion.

We have

$$x_i^p = x_i^{p-1} + \Delta x_i^p + \lambda \zeta_i^p, \tag{33}$$

where again the correction to keep the centroid fixed is neglected, and Δx_i^p denotes the hydrodynamic contribution to the displacement. Then

$$(x_i^p)^2 = (x_i^{p-1} + \Delta x_i^p)^2 + 2\lambda\zeta_i^p(x_i^{p-1} + \Delta x_i^p) + \lambda^2(\zeta_i^p)^2. \tag{34}$$

The hydrodynamics conserves angular momentum, to within a neglegible numerical error; hence summing (34) over i and adding the corresponding term for y_i^p,

$$A^p = A^{p-1} + (\lambda^2/N)\sum_{i=1}^{N}[(\zeta_i^p)^2 + (\eta_i^p)^2]$$

$$+ (2\lambda/N)\sum_{i=1}^{N}\{\zeta_i^p(x_i^{p-1} + \Delta x_i^p) + \eta_i^p(y_i^{p-1} + \Delta y_i^p)\}. \tag{35}$$

Now

$$x_i^q = x_i^0 + \sum_{r=1}^{q}\Delta x_i^r + \lambda\sum_{r=1}^{q}\zeta_i^r. \tag{36}$$

We deduce from (35) that

$$A^p = A_v^p + H^p, \tag{37}$$

where A_v^p, the value due to pure diffusion, can be written as

$$A_v^p = A^0 + (\lambda^2/N)\sum_{q=1}^{p}\sum_{i=1}^{N}[(\zeta_i^q)^2 + (\eta_i^q)^2] + (2\lambda/N)\sum_{q=1}^{N}\sum_{i=1}^{N}[\zeta_i^q x_i^0 + \eta_i^q y_i^0]$$

$$+ (2\lambda^2/N)\sum_{q=1}^{p}\sum_{i=1}^{N}\sum_{r=1}^{q-1}[\zeta_i^q\zeta_i^r + \eta_i^q\eta_i^r], \tag{38}$$

and H^p, the contribution of the hydrodynamic interaction, is given by

$$H^p = (2\lambda/N)\sum_{q=1}^{p}\sum_{i=1}^{N}\sum_{r=1}^{q}\{\zeta_i^q\,\Delta x_i^r + \eta_i^q\,\Delta y_i^r\}. \tag{39}$$

Clearly,

$$E(H^p) = 0, \tag{40}$$

since Δx_i^r for $r \leqslant q$ is independent of ζ_i^q. To estimate the standard deviation of H^p, the variables

$$\sum_{r=1}^{q}\Delta x_i^r, \qquad \sum_{r=1}^{q}\Delta y_i^r$$

must be considered. For the conditions of our calculations, the displacement due to

the noise is small compared with the displacement due to the hydrodynamics. After three rotations, the change in radius is a small fraction of the original value. Thus

$$\sum_{r=1}^{q} \Delta x_i^r \approx \sum_{r=1}^{q} \Delta y_i^r \approx \min\left(\frac{q\Gamma}{4\pi R_0}\Delta t,\ R_0\right). \tag{41}$$

Then for $t \ll T$,

$$H^p \approx \frac{2\lambda}{N} \sum_{q=1}^{p} \sum_{i=1}^{N} q(\zeta_i^q + \eta_i^q) \frac{\Gamma \Delta t}{4\pi R_0}, \tag{42}$$

and for $t \gg T$,

$$H^p = \frac{2\lambda}{N} \sum_{q=1}^{p} \sum_{i=1}^{N} (\zeta_i^q + \eta_i^q) R_0. \tag{43}$$

In the former case, after some rearranging,

$$\mathrm{var}(H^p) = \frac{16\pi^2}{3}\left(\frac{t}{T}\right)^2 \frac{\lambda^2 p A^0}{N}, \tag{44}$$

and in the latter case,

$$\mathrm{var}(H^p) = 16\lambda^2 p A^0/N. \tag{45}$$

If $\sum_{i=1}^{N} \Delta x_i$ is dominated by hydrodynamics, the correlation between H^p and A_ν^p is small, as can be seen by inspection of the terms that arise when (38) and (39) are multiplied together. Thus var(A) is obtained by adding together (29), (44), and (45). The interaction value is thus at most a factor of two greater than for pure diffusion, and is the same for $t \ll T$.

On Fig. 3, we have also plotted $\sigma(\epsilon)$ as given by Eq. (30) for $N = 1000$. The random

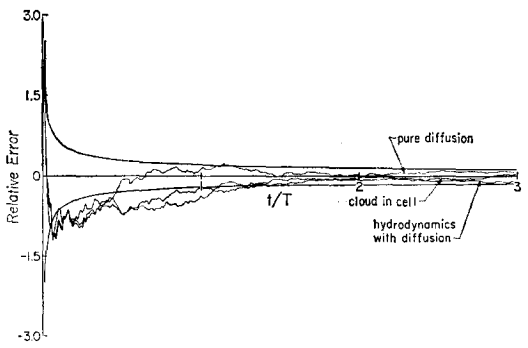

FIG. 3. Relative error vs t ($t < 3T$) for pure diffusion and hydrodynamic motion with diffusion for $N = 1000$, using the method of Section 2 and the cloud in cell method. Also given is $\sigma(\epsilon)$.

values are consistent with the calculated $\sigma(\epsilon)$. To check the ideas further, a long run of up to 19 rotation times was done with 50 vortices, and the results are shown in Fig. 4.

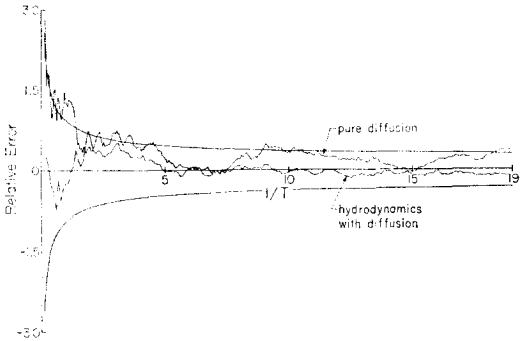

FIG. 4. Relative error vs t ($t < 19T$) for pure diffusion and hydrodynamic motion with diffusion for $N = 50$. Also given is $\sigma(\epsilon)$.

5. CLOUD IN CELL

Since a large number of vortices is desirable to reduce the error, alternative and considerably faster ways of calculating the velocity of the vortices are required. A possible method is the "cloud in cell," described, for example, by Christiansen [8]. For purposes of comparison, we present here results obtained for 1000 point vortices using this approach.

The cloud in cell technique introduces a mesh and replaces each vortex in a cell by four contributions at the mesh points. A fast Poisson solver evaluates the stream function at each mesh point and interpolation gives the velocity at the original vortex, whose position is now updated. There is some choice in the interpolation methods by which the meshpoint contributions to the vorticity are evaluated and the velocity is calculated from the stream function, and in the method of updating the position. We have adopted the following procedure.

The vortex core, initially of radius R_0, is placed inside the rectangular region $[a, b] \times [c, d]$ which is covered by a uniform $I \times J$ net:

$$\begin{aligned} x^i &= a + (i-1)\,h, \qquad h = 1/I, \qquad i = 1,\ldots, I+1, \\ y^j &= c + (j-1)\,k, \qquad k = 1/J, \qquad j = 1,\ldots, J+1. \end{aligned} \tag{46}$$

Using the usual five-point operator to approximate Poisson's equation, relating the streamfunction, Ψ, to the vorticity, ω, on the net (46) gives the difference equations

$$(1/h^2)\{\Psi_{i+1,j} - 2\Psi_{i,j} + \Psi_{i-1,j}\} + (1/k^2)\{\Psi_{i,j+1} - 2\Psi_{i,j} + \Psi_{i,j-1}\} = -\omega_{i,j}\,, \qquad i = 2,\ldots, I, \quad j = 2,\ldots, J. \tag{47}$$

Here $\Psi_{i,j}$ approximates $\Psi(x^i, y^j)$ and $\omega_{i,j} = \omega(x^i, y^j)$. On the boundary of the rectangular region

$$\Psi(x, y) = -(\Gamma/4\pi) \ln(x^2 + y^2). \tag{48}$$

This is the exact boundary condition in the absence of diffusion. If the boundary of the region is far enough away from the vortex core and the time for which the soluiton is computed is small then this is a good approximation.

The solution, $\Psi_{i,j}$, of Eqs. (47) subject to the boundary conditions (48) is computed efficiently using standard fast Poisson solvers [9].

From the streamfunction $\Psi_{i,j}$ the velocity field $(u_{i,j}, v_{i,j})$ is calculated using the centered difference approximations

$$\begin{aligned} u_{i,j} &= (1/2k)(\Psi_{i,j+1} - \Psi_{i,j-1}), \\ v_{i,j} &= -(1/2h)(\Psi_{i+1,j} - \Psi_{i-1,j}). \end{aligned} \tag{49}$$

To obtain $\omega_{i,j}$ from the distribution of point vortices as well as to calculate the velocity at each vortex point from the velocity field $(u_{i,j}, v_{i,j})$, bilinear interpolation or the method of area weighting is used. If the position of the nth vortex is $x_n = x^i + \beta h$, $y_n = y^j + \alpha k$ $0 \leqslant \alpha, \beta < 1$ then the fraction of the total vorticity assigned to each of the four surrounding net points (x^i, y^j), (x^{i+1}, y^j), (x^{i+1}, y^{j+1}), (x^i, y^{j+1}) is given by $F_1 = (1 - \beta)(1 - \alpha)$, $F_2 = \beta(1 - \alpha)$, $F_3 = \alpha\beta$, $F_4 = (1 - \beta)\alpha$, respectively. To evaluate the velocity of the vortex at (x_n, y_n) the same area weighting gives

$$u(x_n, y_n) = F_1 u_{i,j} + F_2 u_{i+1,j} + F_3 u_{i+1,j+1} + F_4 u_{i,j+1},$$

and similar for $v(x_n, y_n)$. This method of interpolating has the advantage of leaving invariant the total circulation as well as the linear impulses $\sum \kappa_i x_i$, $\sum \kappa_i y_i$ within each cell. The error introduced in the angular impulse is second order in the grid size.

In our calculation $a = c = -2$, $b = d = 2$, and $I = J = 64$. Since $R_0 = 1$ and 1000 point vortices are used, this corresponds to a density of 1 vortex per cell within the core. This density is minimal if the vortex core is to be represented adequately.

The point vortices are advanced in time and diffusion is incorporated, as described in Section 2. As in Section 2, $\Delta t = T/100$. With this time step using cloud in cell together with modified Euler, it is estimated that the error in A is approximately 1.5% of $4\nu t$.

The advantages of the method are that the number of computations is roughly linear in the number of vortices and mesh points. Thus many more vortices can be handled than in the method described in Section 2. Also, the finite size of the vortex is irrelevant, but it is replaced, in effect, by the cell size which has an effect on the computations that has not yet been clarified.

Figure 3 shows results for $N = 1000$ and t up to $3T$. The computation time required using the cloud in cell method was roughly $\frac{1}{20}$ the time necessary using the method of Section 2.

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